33. In Phenomenological Landau theory for phase transitions the (Landau-) free energy is an analytic function of the order parameter $\eta$

$$\mathcal{L} = \frac{L}{V} = \sum_{n=0}^{\infty} a_n([K], T) \eta^n,$$

where the coefficients $a_n([K], T)$ depend on the coupling constants $[K]$ and temperature. The order parameter distinguishes the different phases: in the disorder phase it is zero $\eta \equiv 0$ and in the ordered phase it has a finite non-zero value and it acquires the value that minimizes the free energy $\mathcal{L}$. The free energy must be consistent with the symmetries of the system. Near transition point $\eta \ll 1$ so that only lowest order terms $n \leq 4$ must be included and the coefficients $a_n([K], T)$ can be expanded as power series in $t \equiv (T - T_c)/T_c$.

A.) For the Ising model the order parameter is the average magnetization $M$. Show that the Landau theory produces the mean-field theory for the Ising model, when the order parameter has only one component.

B.) Write down a Landau free energy for a magnetic system for the case where the magnetization is a 3 dimensional vector $\vec{m} = (m_x, m_y, m_z)$.

C.) Show that if the symmetry requirement $\mathcal{L}(\eta) = \mathcal{L}(-\eta)$ is relaxed, the transition is a first order transition i.e. the order parameter jumps discontinuously from $\eta = 0$ to $\eta = \eta_c$ at the transition point.

34. For inhomogeneous systems $\eta = \eta(\vec{r})$ the (Landau-)free energy density $L$ is an integral over the whole space (Landau free energy becomes a functional $L[\eta(\vec{r})]$). Variations in the order parameter profile increase the energy; consequently the free energy density has a term proportional to the gradient $\nabla \eta$

$$L[\eta(\vec{r})] = \int d^d\vec{r} \frac{\gamma}{2} (\nabla \eta(\vec{r}))^2 + a_2 \eta^2(\vec{r}) + a_4 \eta^4(\vec{r}) - H(\vec{r}) \eta(\vec{r})$$,

where the coefficients $a_n([K], T)$ are the same as in the homogeneous Landau theory and $d$ is the dimension. The mean field approximation for the average profile $\langle \eta(\vec{r}) \rangle$ follows from the Landau free energy $L$ again through maximization, i.e. the functional derivative of $L$ must be zero

$$\frac{\delta L[\eta(\vec{r})]}{\delta \eta(\vec{r})} \bigg|_{\eta = \langle \eta \rangle} = 0.$$
Show that for the correlation function

$$G(\vec{r}; \vec{r}') \equiv \langle \eta(\vec{r}) \eta(\vec{r}') \rangle - \langle \eta(\vec{r}) \rangle \langle \eta(\vec{r}') \rangle$$

applies

$$G(\vec{r}; \vec{r}') = kT \frac{\delta \langle \eta(\vec{r}) \rangle}{\delta H(\vec{r})}$$

The averages are calculated as

$$\langle \eta(\vec{r}) \rangle = \frac{Tr[\eta(\vec{r}) e^{-\beta L[\eta(\vec{r})]}]}{Tr[e^{-\beta L[\eta(\vec{r})]}]} \quad ; \quad Tr[e^{-\beta L[\eta(\vec{r})]}] \equiv \int D[\eta] e^{-\beta L[\eta(\vec{r})]},$$

where the integration is over all profiles $\eta(\vec{r})$.

Derive the equation for the correlation function by inserting the equation following from the minimization requirement (3) into (5). Show that the correlation function can be written as

$$G(\vec{r}, 0) = C|\vec{r}|^{(2-d)} Y(|\vec{r}|/\xi),$$

where $d$ is the dimension of the system, $\xi(t)$ the correlation length and $Y(x)$ is some function (i.e. $Y(|\vec{r}|/\xi)$ depends only on the ratio of $|\vec{r}|$ and the correlation length $\xi$).

How does $\xi$ depend on the temperature near the transition temperature $t = 0$?

35. To study the validity of the mean field theory one must approximate the strength of fluctuations $|\langle \eta(\vec{r}) \eta(\vec{r}') \rangle - \langle \eta(\vec{r}) \rangle \langle \eta(\vec{r}') \rangle| = |G(\vec{r}, \vec{r}')|$ neglected in the theory. Below the transition temperature this can be done by comparing the fluctuations to $\langle \eta(\vec{r}) \rangle^2$. The fluctuations should be averaged over a region of size $\xi^d$, where they are correlated. A quantitative measure for the importance of fluctuations is given by ratio

$$R = \frac{\int_{\Omega(\xi)} d^d r |G(\vec{r}, 0)|}{\int_{\Omega(\xi)} d^d r \langle \eta(\vec{r}) \rangle^2},$$

where $\Omega(\xi)$ is a volume of the region of linear size $\xi$. If $R \ll 1$ the mean field theory is valid.

Using the results obtained so far show that for the Ising model the mean field theory is valid at the transition point $t \to 0$ for $d > d_c$ and breaks down for $d < d_c$, where $d_c$ is so called upper critical dimension. What is $d_c$?

Hint: You don’t have to calculate the integrals, the dependence on $\xi$ can be scaled out. And a reminder: $\langle \eta(\vec{r}) \rangle \sim |t|^{1/2}$ for the Ising model.

36. Prove that the cross-correlation function of $\vec{v}$ and $\vec{F}$ of a Brownian particle is

$$\langle \vec{v}(t) \vec{F}(t) \rangle = \frac{3kT}{mB}$$

Langevin equation:

$$m \frac{d\vec{v}(t)}{dt} = -\frac{\vec{v}(t)}{B} + \vec{F}(t)$$